Exercise 4

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

$$y'' - 6y' + 8y = 1, \; y(0) = 1, \; y'(0) = 1$$

Solution

Let

$$y''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\begin{aligned} \int_0^x y''(t)\,dt &= \int_0^x u(t)\,dt \\ y'(x) - y'(0) &= \int_0^x u(t)\,dt \end{aligned}$$

Substitute $y'(0) = 1$ and bring it to the right side.

$$y'(x) = 1 + \int_0^x u(t)\,dt \tag{2}$$

Integrate both sides again from 0 to x.

$$\begin{aligned} \int_0^x y'(s)\,ds &= \int_0^x \left[1 + \int_0^s u(t)\,dt\right] ds \\ y(x) - y(0) &= x + \int_0^x \int_0^s u(t)\,dt\,ds \end{aligned}$$

Substitute $y(0) = 1$ and bring it to the right side.

$$y(x) = 1 + x + \int_0^x \int_0^s u(t)\,dt\,ds$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^s u(t)\,dt & dw &= ds \\ dv &= u(s)\,ds & w &= s \end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned} y(x) &= 1 + x + s\int_0^s u(t)\,dt\bigg|_0^x - \int_0^x su(s)\,ds \\ &= 1 + x + x\int_0^x u(t)\,dt - \int_0^x su(s)\,ds \\ &= 1 + x + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\ &= 1 + x + \int_0^x (x-t)u(t)\,dt \end{aligned} \tag{3}$$

Substitute equations (1), (2), and (3) into the original ODE.

$$y'' - 6y' + 8y = 1 \quad \rightarrow \quad u(x) - 6\left[1 + \int_0^x u(t)\,dt\right] + 8\left[1 + x + \int_0^x (x-t)u(t)\,dt\right] = 1$$

Expand the left side.

$$u(x) - 6 - 6\int_0^x u(t)\,dt + 8 + 8x + 8\int_0^x (x-t)u(t)\,dt = 1$$

$$u(x) + 2 + 8x + \int_0^x (-6)u(t)\,dt + \int_0^x 8(x-t)u(t)\,dt = 1$$

$$u(x) + 2 + 8x + \int_0^x [-6 + 8(x-t)]u(t)\,dt = 1$$

$$u(x) = -1 - 8x - \int_0^x [-6 + 8(x-t)]u(t)\,dt$$

Therefore, the equivalent Volterra integral equation is

$$u(x) = -1 - 8x + \int_0^x [6 - 8(x-t)]u(t)\,dt.$$